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# POINCARÉ TRANSVERSALITY AND FOUR-DIMENSIONAL SURGERY

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THIS NOTE gives a condition, condition-$\pi$, on a four-dimensional surgery problem which guarantees the existence of a topological solution. This criteria is then applied to the fundamental or "atomic surgery" problems, $M^4 \to X$. It is seen that these satisfy condition-$\pi$ iff a fairly weak transversality condition holds for the map classifying the fundamental group: $X \to \vee S^1$. Combining these two observations, we see that the topological surgery "theorem" holds in dimension four iff a certain problem in homotopy theory can be solved.

We consider surgery problems in the sense of Wall [6] $(M, \partial) \xrightarrow{f} (X, \partial)$ which are $Z\pi_1(X)$-equivalences over $\partial X$ and have a well defined—and vanishing—obstruction $\theta$ in $L^s$ or $L^h$ $(\pi_1(X))$. Also we presume low dimensional surgery has been done so that $f$ induces an isomorphism on $\pi_1$ and $\ker_2(f)$ is a direct sum of standard planes. In this case, we may let $h: \coprod S^2 \vee S^2 \to M^4$ represent $k_2(f)$. We say that $h$ satisfies *condition-$\pi$* if every loop in $h(\coprod S^2 \vee S^2)$ is null homotopic in $M$. (In other words the double points of $h$ do not wrap around the fundamental group of $M$.) Similarly, $f$ satisfies *condition-$\pi$* if it is normally bordant (rel $\partial$) to an $f'$ such that $k_2(f')$ is represented by some $h'$ satisfying condition-$\pi$. We prove:

THEOREM 1. *If $f$ satisfies condition $\pi$ then $f$ is normally cobordant (rel $\partial$) to a (simple in the case of $\theta = 0 \in L^s$) homotopy équivalence.*

*Remark.* If we further assumed $\pi_1(M^4 - h(\coprod S^2 \vee S^2)) \xrightarrow{\text{inc}_\#} \pi_1(M^4)$ to be an isomorphism then the theorem would follow from earlier results on doubles of boundary links [2]. Unfortunately, this strengthened hypothesis is often difficult to achieve in practice. The reason is that the finger moves used to reduce the kernel of $\text{inc}_\#$ introduce loops in $h(\coprod S^2 \vee S^2)$ exactly of the sort we wish to exclude.

*Proof.* Put $h$ in general position. Let $\mathcal{N} = \mathcal{N}(h(S^2 \vee S^2))$ be the regular neighborhood. Without loss of generality, we assume $\mathcal{N}$ is connected. As in [2] we will describe a typical link $L$ such that zero framed surgery on $L$, $\mathcal{S}(L)$, is diffeomorphic to $\partial \mathcal{N}$.

We will build a bordism in four steps. The first step is to form $W_1 = \mathcal{S}(L) \times [0, 1]$. The second step is to attach certain "generalized kinky handles" to $\mathcal{S}(L) \times 1$ to form $W_2$. The third step forms $W_3$ by attaching certain "relation" 2-handles to $\mathcal{S}(L) \times 0$. The final step deletes imbedded kinky handles in $(W_3, \partial_0 W_3)$ which yields $W_4$. The general plan is, as in [2], to produce $(W_4; \partial_1, \partial_0)$ with certain key properties which enable surgery to be completed.

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The necessary homological (and fundamental group) properties of $W_4$ are given in Lemma 3 of [2] (substitute $W_4$ for $Q$), to which the reader should refer. The argument there will not be reproduced but modified.

The lower boundary $\partial_0 W_4$ is required to be zero framed surgery on a "good boundary link" $L$ (with associated surgery problem $P$) and $\partial_1 W_4$ is equal to $\partial(\mathcal{N}^+)$ where $\mathcal{N}^+$ is a regular neighborhood of the 2-complex $K$ formed from $h(\coprod S^2 \vee S^2)$ by attaching immersed (general position) null homotopies to a basis for $\pi_1(h(\coprod S^2 \vee S^2))$. It is required $W_4$ should be $Z\pi_1$-homology equivalent to a four-dimensional one-handle body with the interior of another null homotopic one-handle body deleted from its interior. Cutting out $\mathcal{N}^+$ and giving in $W_4 \cup P$ concentrates the surgery problem away from the fundamental group and leads to the solution.

The main difference between the present setting and [2] is that when $K$ is written $K = h(\coprod S^2 \vee S^2) \cup \Delta$'s the 2-cells $\Delta$ may have interior intersections with $h(\coprod S^2 \vee S^2)$. This means that $\mathcal{N}^+$ is not obtained from $\mathcal{N}$ by attaching kinky-handles but rather by attaching objects of the form $(\Delta - \coprod \delta\text{'s}) \times D^2$/self-plumbings, where $\delta$'s are interiors of disjoint closed subdisks in interior $(\Delta)$. That is, we attach "kinky planar surfaces."

In what follows link ramifications do not seriously affect the argument. In Figs 1–5, we assume $h(\coprod S^2 \vee S^2)$ has the form:

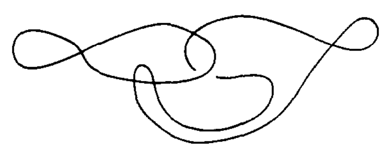

Fig. 1.

That is, $\coprod S^2 \vee S^2 = S^2 \vee S^2$ and $h$ introduces one double point into each sphere, and one pair of intersection points between the spheres. From Kirby calculus we calculate a description for $L$.

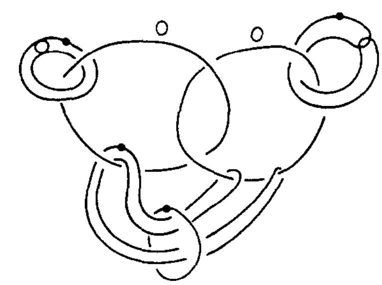


Fig. 2.

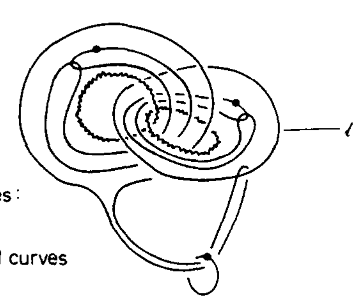


Fig. 3.

Note: clasp parity is not important and not distinguished. Also one component $\ell$ of $L$ is not drawn as a circle but as a wedge of two circles. This is the spine of a Siefert surface for $\ell$. To obtain $\ell$ plumb to untwisted bands along the wedge. These conventions simplify manipulations.

The boundaries of the disks $\{\delta\text{'s}\}$ attach to the small circles linking the 2-handles in Fig. 2. These become the serrated curves $a$ and $b$ in Fig. 3. It is easy to see that $L$ is a boundary link, $L=\partial S$. There is a rather natural choice of S visible in Fig. 3; choose this $S$. Let $\{\gamma\text{'s}\}$ be a collection of simple loops representing a simplectic basis of $S$. One may check that the curves: $a$, $b$, $\gamma$'s are disjoint or may be perturbed to be disjoint from $S$. It follows that

$$[a],\ [b],\ [\gamma\text{'s}]\in\pi_1(S^3-L)_\omega,$$

where $\omega$ denotes the intersection of the finite lower central series. As in [2] handles may be attached along a restricted class of curves: $\hat{r}\text{'s}\subset\mathcal{S}(L)\times 0$ to kill $[a]$, $[b]$, and $[\gamma\text{'s}]$. The restriction to this class is necessary to achieve the required properties of the lower boundary $\partial_0 W_4$.

Consider the analog of $k^0$ (p. 460 [2]). In our setting $k^0$ will have three pieces: product collar = meridian loop × $I$, a planar surface, and caps deployed as below.

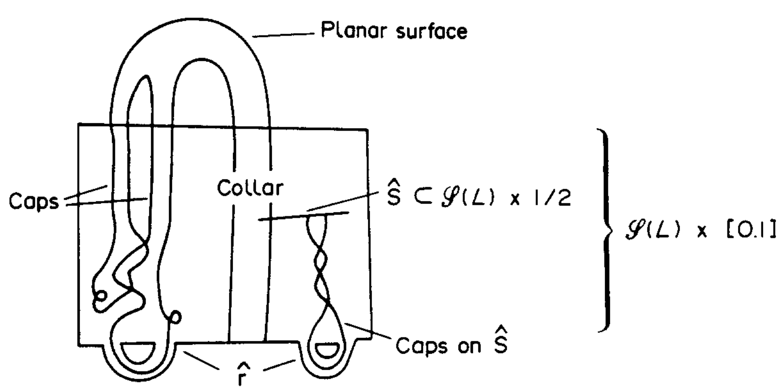


Fig. 4.

Clearly there is a potential (not drawn) for the caps on the planar surface to intersect the caps on $S$ (although the former are easily arranged to be disjoint from $S$ itself). This will be a problem since we wish to construct several (disjoint) geometrically dual spheres to the $k^0$'s.

At this point we normalize the construction of $\mathcal{N}^+$ from $\mathcal{N}$ (using the spinning trick [3]) so that the homological framing of the $\Delta$'s w.r.t. Fig. 3 is zero. Let $n$ be the total number of disks $\mathcal{S}$ in the description of the planar surfaces, and let $m$ be the total number of self-plumbings (kinks).

In the level $\mathcal{S}(L)\times 1/2$, let $S$ denote the completion of $\hat{S}$ to a closed surface. In that level take $[n(n+1)/2]+m+2$ parallel copies of $\hat{S}$. Find caps in $[\mathcal{S}(L)\times[0,1/2]\bigcup_{\hat{r}\text{'s}}$ two-handles] for these surfaces. Using the "capped-surface trick," the copies of $S$ may be transformed into a collection of $[(n(n+1))/2]+m+2$ disjoint immersed dual spheres to each $k^0$. All intersections (and by this we include all self intersections) of the $k^0$'s may be oriented (in one of two ways) and then classified according to their associated fundamental group elements in $\pi_1(W_3)$. There need be no more than $[(n(n+1))/2]+m$ classes. To each class of intersection and each $k^0$ assign a fixed dual. Remove intersections in this class by piping to a parallel copy of the assigned dual. The results are $\{k^1\text{'s}\}$. It may be checked that all the double points in this collection have the trivial associated element in $\pi_1(W_3)$. Tubing into copies of one of the two remaining duals will restore the attaching framing to zero (call the results $k^2$'s). The final dual exhibits the (actually unnecessary, but technically convenient) condition $\pi_1(W_3 \to \cup(k^2)) \to \pi_1(W_3)$ is an isomorphism.

Form $W_4 = \overline{W_3 - \cup(k^2)}$. That $W_4$ meets its design specification is similar to the verification in [2]. We only note that the link $L''$ (with $(L'') = \partial_0 W_4$) is obtained from Fig. 3 by: (1) adding the curves $\{f\text{'s}\}$; (2) ramified Whitehead doubling of the original components; and (3) adding commutator curves formed by plumbing pairs of untwisted bands centered along parallel copies of the original components; thus $L''$ is a $g$-$b$-link. This completes the proof of Theorem 1.

The atomic surgery problems (see [2] for references) all have targets of the form $(X, \partial X)$ where $X$ is a mapping cylinder $M(N^3 \xrightarrow{\alpha} \vee S^1)$ from a closed three-manifold $N^3$ to a wedge of circles. The map $\alpha$ must satisfy the conditions: (1) $\alpha_\#$ is onto, and (2) ker $\alpha_\#$ is a perfect group. Furthermore, the atomic problems are diffeomorphisms over $\partial X \cong N^3$. If such a problem has a solution: $(M, N^3) \underset{e}{\overset{\cong}{\to}} (X, N^3)$ (where $k$ is a homotopy equivalence equal to the identity on the boundary), then by topological transversality ([4]) the composition $M \underset{e}{\to} X \to \vee S^1$ may be perturbed to be transverse (relative to any map transverse on $\partial M = N^3$) to the collection of points $\{P_i\} \subset \vee S^1$ opposite the base point in each circle summand. Thus the surgery "theorem" implies a strong form of "Poincaré transversality" for $X \to \vee S^1$.

Conversely we will show that a weak form of Poincaré transversality implies the surgery theorem. Curiously this implies that a weak transversality is automatically promoted to a much stronger type. However, the main point of this observation is that the surgery "theorem" is equivalent to a homotopy theoretic question—unfortunately not an easy one.

Let $(Y, \Sigma)$ be a C.W. complex with finitely generated fundamental group with $\Sigma$ an orientable closed surface. We say $(Y, \Sigma)$ is a three-dimensional *Z-Lefschetz duality pair* (3-$Z$-LDP) if there is a homology class $\mu \in H_3(Y, \Sigma; Z)$ so that the cup product with $\mu$ induces isomorphisms:

$$H^*(Y, \Sigma; Z) \to H_{3-*}(Y; Z) \text{ and } H^*(Y; Z) \to H_{3-*}(Y, \Sigma; Z).$$

If $\Sigma = \phi$. $Y$ is called a Z-Poincaré duality space (3-$Z$-PDS). Similarly a C.W. pair $(Z, Y)$ is a (4-$Z$-LDP) if $\pi_1(Z)$ is finitely generated, $Y$ is a (3-$Z$-PDS) and there exists $\mu \in H_4(Z, Y; Z)$ inducing Lefschetz duality as above.

We say that $(X) \to (\vee S^1)$ is *weakly transverse* to $\{P_i\}$ if there is an $(X', N^3)$ homotopy equivalent (rel *id* on $\partial$) to $(X, N^3)$ and $X'$ can be decomposed as: $X' = H \cup (\coprod_i Y_i \times I)$ with the unions taking place over $\coprod_i Y_i \times \{0, 1\}$ and the induced inclusions $\Sigma_i \subset \partial X' = N^3$ being the inverse images of $\{P_i\}$ under some (transverse) map homotopic to $\alpha$. The pair $(H, \partial H)$ is required to be a (4-$Z$-LDP) and $(Y_i, \Sigma_i)$ a (3-$Z$-LDP). (These requirements are not actually independent, the first follows from the second.)

THEOREM 2. *Let $f$: $(M, N) \to (X, N)$ be an atomic surgery problem. Suppose $X \xrightarrow{\alpha} \vee S^1$ is weakly transverse to $\{P_i\}$ then $f$ is normally cobordant (rel $\partial$) to a homotopy equivalence.*

*Proof.* We show that $f$ is normally cobordant (rel $\partial$) to $f'$ satisfying condition-$\pi$. Theorem 1 then applies. Since the Cappell–Shaneson $\Gamma$-group $\Gamma_3(Z[\pi_1 Y_i] \to Z) \cong L_3(\mathrm{e})$

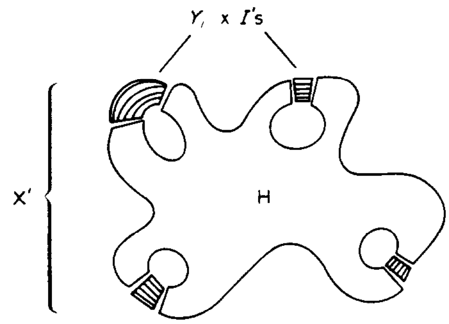


Fig. 5.

$\cong 0$ (see [1]) we may normally bord $f$ (rel $\partial$) to an integral homology equivalence over $\coprod_i (Y_i \times I)$. (Details of this argument in the low dimensional case are also given by Turaev [5]). Now cut open to get an integral homology surgery problem $g$ over $H$. (By Myer–Vietoris we have, at this point, a $Z$-homology equivalence over $\partial H$.) Perform zero- and one-surgeries to make $g_\#$ an isomorphism. A homological argument shows that the signature of the (singular) intersection form on $H$ is equal to signature $(M) = \theta(f) +$ signature $X = 0$. The normal data implies that the intersection form on $k_2(g; Z)$ is even. Thus by the classification of nonsingular quadratic forms $k_2(g; Z)$ is a sum of standard planes $\oplus \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$ over the integers. Since $g_\#$ is an isomorphism $k_2(g; Z)$ is spherical. Thus a basis for the standard planes is represented by a map $h(\coprod (S^2 \vee S^2))$.

This basis for the integral kernel of $g$ becomes a $Z[\pi_1 X]$-basis for the kernel of the surgery problem $f': M' \to X$ obtained by regluing $H$. Since $\pi_1(H) \xrightarrow{\text{inc}_\#} \pi_1(X)$ is the zero map, this basis satisfies condition-$\pi$, and the proof is complete.

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